

NASA/CR-2002-211923
ICASE Report No. 2002-33



Termination of String Rewriting Rules that have One Pair of Overlaps

Alfons Geser
ICASE, Hampton, Virginia

DISTRIBUTION STATEMENT A
Approved for Public Release
Distribution Unlimited



October 2002

20021129 057

The NASA STI Program Office . . . in Profile

Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA Scientific and Technical Information (STI) Program Office plays a key part in helping NASA maintain this important role.

The NASA STI Program Office is operated by Langley Research Center, the lead center for NASA's scientific and technical information. The NASA STI Program Office provides access to the NASA STI Database, the largest collection of aeronautical and space science STI in the world. The Program Office is also NASA's institutional mechanism for disseminating the results of its research and development activities. These results are published by NASA in the NASA STI Report Series, which includes the following report types:

- **TECHNICAL PUBLICATION.** Reports of completed research or a major significant phase of research that present the results of NASA programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA's counterpart of peer-reviewed formal professional papers, but having less stringent limitations on manuscript length and extent of graphic presentations.
- **TECHNICAL MEMORANDUM.** Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.
- **CONTRACTOR REPORT.** Scientific and technical findings by NASA-sponsored contractors and grantees.

- **CONFERENCE PUBLICATIONS.** Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or cosponsored by NASA.
- **SPECIAL PUBLICATION.** Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.
- **TECHNICAL TRANSLATION.** English-language translations of foreign scientific and technical material pertinent to NASA's mission.

Specialized services that complement the STI Program Office's diverse offerings include creating custom thesauri, building customized data bases, organizing and publishing research results . . . even providing videos.

For more information about the NASA STI Program Office, see the following:

- Access the NASA STI Program Home Page at <http://www.sti.nasa.gov>
- Email your question via the Internet to help@sti.nasa.gov
- Fax your question to the NASA STI Help Desk at (301) 621-0134
- Telephone the NASA STI Help Desk at (301) 621-0390
- Write to:
NASA STI Help Desk
NASA Center for AeroSpace Information
7121 Standard Drive
Hanover, MD 21076-1320

NASA/CR-2002-211923
ICASE Report No. 2002-33



Termination of String Rewriting Rules that have One Pair of Overlaps

Alfons Geser
ICASE, Hampton, Virginia

ICASE
NASA Langley Research Center
Hampton, Virginia

Operated by Universities Space Research Association



Prepared for Langley Research Center
under Contract NAS1-97046

October 2002

Available from the following:

NASA Center for AeroSpace Information (CASI)
7121 Standard Drive
Hanover, MD 21076-1320
(301) 621-0390

National Technical Information Service (NTIS)
5285 Port Royal Road
Springfield, VA 22161-2171
(703) 487-4650

TERMINATION OF STRING REWRITING RULES THAT HAVE ONE PAIR OF OVERLAPS*

ALFONS GESER[†]

Abstract. This paper presents a partial solution to the long standing open problem of termination of one-rule string rewriting. Overlaps between the two sides of the rule play a central role in existing termination criteria. We characterize termination of all one-rule string rewriting systems that have one such overlap at either end. This both completes a result of Kurth and generalizes a result of Shikishima-Tsuji et al.

Key words. semi-Thue system, string rewriting, one-rule, single-rule, termination, uniform termination, overlap

Subject classification. Computer Science

1. Introduction and Related Work. Termination of one-rule string rewriting systems (SRSs) is a long standing open problem [12, 13, 11, 15, 14, 7, 16, 18, 2, 3, 4]. The first systematic approach was started by Kurth [8]. He introduced a number of termination criteria to solve termination for all $\ell \rightarrow r$ where $|r| \leq 6$.¹

Most of Kurth's criteria (5 out of 8), and indeed most of the criteria introduced since, are based on two sets: the set of overlaps of the left hand side (from the left end) with the right hand side (from the right end); and the set of overlaps of the right hand side (from the left end) with the left hand side (from the right end). Kurth's Criterion D states that we have termination if one or both of the two sets are empty.

In the case where both sets are singletons, we say that the one-rule SRS has *one pair of overlaps*. Kurth [8] provides Criterion F specifically for this case. As Criterion F can only prove termination of rules that are left barren or right barren, it is incomplete as we will show (Example 2). Shikishima-Tsuji et al. [16, Theorem 2] show that a *confluent* one-rule SRS with one pair of overlaps terminates if and only if there are no loops of lengths 1 or 2. As a consequence termination of such SRSs is decidable.

This paper completely solves the termination problem for one-rule SRSs with one overlap pair. We prove that such an SRS terminates if and only if it has no loop of lengths 1, 2 or 3 (Theorem 7.1). This implies decidability of the termination problem.

It turns out that the extension is non-trivial. There are two behaviours that were observed neither by Kurth nor by Shikishima-Tsuji et al. Loops of length 3 is one of them; the other is terminating non-tame rules.

This paper makes the following original contributions:

1. Termination of one-rule SRSs with one overlap pair is shown decidable.
2. Termination of one-rule SRSs with one overlap pair is shown equivalent to the non-existence of loops of length 3 or less.
3. Terminating one-rule SRSs with one overlap pair are shown to have linear derivation lengths.
4. The first termination criterion for a class of non-tame one-rule SRSs.

*This work was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-97046 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23681-2199, USA.

[†]Address: ICASE, Mail Stop 132C, NASA Langley Research Center, Hampton, VA 23681. E-mail: geser@icase.edu

¹An English presentation of Kurth's chapter on termination can be found in the author's habilitation thesis [3].

The paper is organized as follows. After the preliminaries (Section 2) and an introduction to left barren and tame rules (Section 3), we focus on the interesting non-tame case. In Section 4, we derive a pattern that describes the non-tame rules. In Sections 5 and 6, we solve the non-terminating and terminating non-tame rules, respectively. Section 7 finally shows the main theorem of the paper and its ramifications.

2. Preliminaries. A *string rewriting rule* is a pair $\ell \rightarrow r$ of strings, $\ell, r \in \Sigma^*$ where Σ is a given alphabet. A set of string rewriting rules is called a *string rewriting system* (SRS). An SRS R induces a *rewrite step* relation \rightarrow defined by $s \rightarrow t$ if there are $u, v \in \Sigma^*$ and a rule $\ell \rightarrow r$ in R such that $s = u\ell v$ and $t = urv$. The SRS R is said to *terminate* if there is no infinite sequence of rewrite steps $s_1 \rightarrow s_2 \rightarrow \dots$.

A string u is called a *factor* of v if $v = sut$ for some $s, t \in \Sigma^*$; a *prefix* if $v = ut$ for some $t \in \Sigma^*$; a *suffix* if $v = su$ for some $s \in \Sigma^*$. The prefix or suffix u of v is called *proper* if $u \neq v$. The set of *overlaps* of a string u with a string v is defined by

$$\text{OVL}(u, v) = \{w \in \Sigma^+ \mid u = u'w, v = wv', u'v' \neq \varepsilon, u', v' \in \Sigma^*\}.$$

The length of a string u is denoted by $|u|$.

3. Left Barren Rules. For a fixed one-rule SRS $\{\ell \rightarrow r\}$ let $A = \text{OVL}(r, \ell)$ and $B = \text{OVL}(\ell, r)$. In what follows we consider A and B as disjoint. For all $\alpha \in A$, the strings ℓ_α and r_α are defined by $\ell = \alpha\ell_\alpha$ and $r = r_\alpha\alpha$, respectively. Likewise, for all $\beta \in B$, the strings ℓ_β and r_β are defined by $\ell = \ell_\beta\beta$ and $r = \beta r_\beta$, respectively.

The following definition of “left barren” is after McNaughton’s corrected version. The original definition is renamed to “left s-barren” (see Definition 3.4), following a suggestion of Kobayashi et al. [7].

DEFINITION 3.1 (Left barren, right barren [12]). A one-rule SRS $\{\ell \rightarrow r\}$ is called *left barren* if ℓ is not a factor of r and no $\ell_\alpha, \alpha \in A$ is a prefix of any concatenation $r_{\beta_1} \dots r_{\beta_k}$ where $\beta_1, \dots, \beta_k \in B, k \geq 1$. Dually, $\{\ell \rightarrow r\}$ is called *right barren* if ℓ is not a factor of r and no $\ell_\beta, \beta \in B$ is a suffix of any concatenation $r_{\alpha_1} \dots r_{\alpha_k}$ where $\alpha_1, \dots, \alpha_k \in A, k \geq 1$.

A one-rule SRS $\{\ell \rightarrow r\}$ is called *non-overlapping* if $\text{OVL}(\ell, \ell) = \emptyset$.

THEOREM 3.2 ([12]). Every non-overlapping, left barren, one-rule SRS terminates.

THEOREM 3.3 ([3]). Every left barren one-rule SRS terminates.

By symmetry w.r.t. reversal of strings also every right barren one-rule SRS terminates.

DEFINITION 3.4 (Left s-barren, right s-barren [12, 7]). A rule $\ell \rightarrow r$ is called *left s-barren* if no $\ell_\alpha, \alpha \in A$ is a prefix of any $r_\beta, \beta \in B$. Dually $\ell \rightarrow r$ is called *right s-barren* if no $\ell_\beta, \beta \in B$ is a suffix of any $r_\alpha, \alpha \in A$.

A left barren rule is left s-barren, but the converse usually does not hold. Indeed we will encounter left s-barren, not left barren rules later in this paper. They belong to a class of rules whose termination is particularly difficult to show. Next we will define this class.

In the following definition we consider A, B as (disjoint) alphabets. For $\bar{\alpha} = \alpha_1\alpha_2 \dots \alpha_k \in A^*$ we define $\ell_{\bar{\alpha}}$ by $\ell_{\bar{\alpha}} = \ell_{\alpha_1}\ell_{\alpha_2} \dots \ell_{\alpha_k}$. And dually, for $\bar{\beta} = \beta_1\beta_2 \dots \beta_k \in B^*$ we define $\ell_{\bar{\beta}}$ by $\ell_{\bar{\beta}} = \ell_{\beta_1}\ell_{\beta_2} \dots \ell_{\beta_k}$.

Kobayashi et al. [7] introduced the notion of tame, non-overlapping one-rule SRSs.

DEFINITION 3.5 (Tame [3]). Let $\{\ell \rightarrow r\}$ be a one-rule SRS. The sets C and D are defined by

$$\begin{aligned} C &= \{r' \in \Sigma^* \mid r = \beta\ell_{\bar{\alpha}}r', \beta \in B, \bar{\alpha} \in A^*\}, \\ D &= \{r' \in \Sigma^* \mid r = r'\ell_{\bar{\beta}}\alpha, \alpha \in A, \bar{\beta} \in B^*\}. \end{aligned}$$

Then $\ell \rightarrow r$ is called *tame* if ℓ is neither of the form

$$\alpha r_1 r_2 \dots r_k w, \tag{3.1}$$

for any $\alpha \in A$, $k \geq 1$, $r_1, \dots, r_k \in C$, and non-empty prefix w of an element of C ; nor of the form

$$wr_1r_2 \dots r_j\beta, \quad (3.2)$$

for any $\beta \in B$, $j \geq 1$, $r_1, \dots, r_j \in D$, and non-empty suffix w of an element of D .

The following result is implicit in Kobayashi et al. [7, Cor. 5.9].

THEOREM 3.6. *Every non-overlapping, tame, left s-barren one-rule SRS is left barren.*

THEOREM 3.7 ([3]). *Every tame, left s-barren one-rule SRS is left barren.*

By symmetry, every tame, right s-barren one-rule SRS is right barren.

Proof. For a proof by contradiction, assume that $\ell \rightarrow r$ is not left barren, i.e., some ℓ_α is a prefix of some concatenation $r_{\beta_1}r_{\beta_2} \dots r_{\beta_n}$. Let n be minimal. If $n = 1$ then $\ell \rightarrow r$ is not left s-barren. So $n \geq 2$ whence ℓ_α is of the form $r_{\beta_1}r_{\beta_2} \dots r_{\beta_{n-1}}w$ where w is a nonempty prefix of r_{β_n} . Hence ℓ is of the form (3.1) and so $\ell \rightarrow r$ is not tame. \square

4. A Reduction of the Problem. Throughout the remainder of this paper we assume a one-rule SRS $\{\ell \rightarrow r\}$ that has one pair of overlaps, i.e., $|\text{OVL}(r, \ell)| = |\text{OVL}(\ell, r)| = 1$. Let then $\alpha, \beta \in \Sigma^+$ be defined by $\text{OVL}(r, \ell) = \{\alpha\}$ and $\text{OVL}(\ell, r) = \{\beta\}$.

We will devote the greater part of the paper to solving the interesting case: rules that are left s-barren but neither left barren nor right s-barren. According to Theorem 3.7, these are non-tame, specifically they are of the form (3.1). In this section we will derive the general pattern of such rules. Let us henceforth assume that ℓ is not a factor of r and that $|\ell| < |r|$.

The first pattern is derived without the right-s-barren hypothesis.

LEMMA 4.1. *Let $\ell \rightarrow r$ be left s-barren but not left barren. Then $|\beta| > |\alpha|$ and $\ell \rightarrow r$ is of the form*

$$\alpha(ww')^{n-1}w \rightarrow \beta ww' \quad (4.1)$$

for some $n \geq 2$, $w' \in \Sigma^*$, and $w \in \Sigma^+$.

Proof. Let $\ell \rightarrow r$ be left s-barren but not left barren. Then we get by the respective definitions that ℓ_α is not a prefix of r_β and that ℓ_α is a prefix of r_β^n for some $n \geq 1$. Hence r_β is a proper prefix of ℓ_α . So let $\ell_\alpha = r_\beta^{n-1}w$ where $n \geq 2$, and w is a non-empty prefix of r_β . Let $w' \in \Sigma^*$ be defined by $r_\beta = ww'$. By back-substitution we get the form (4.1). From $|\beta r_\beta| = |r| > |\ell| = |\alpha r_\beta^{n-1}w|$ we conclude $|\beta| > |\alpha|$. \square

If we add the right-s-barren hypothesis, then we can rule out the case where α and β overlap in ℓ .

LEMMA 4.2. *If $\ell \rightarrow r$ is left s-barren but neither left barren nor right s-barren, then $|\alpha| + |\beta| \leq |\ell|$.*

Proof. For a proof by contradiction assume $|\alpha| + |\beta| > |\ell|$. Let $\ell \rightarrow r$ be left s-barren but not left barren. By Lemma 4.1 we get that $\ell \rightarrow r$ has the form (4.1). Then by $|\alpha| + |\beta| > |\ell|$ there is a non-empty suffix u of α such that $\beta = u(ww')^{n-1}w$. Define $\alpha' \in \Sigma^*$ by $\alpha = \alpha'u$. The string α' is non-empty by $\beta \neq \ell$. Thus ℓ and r are of the form

$$\begin{aligned} \ell &= \alpha'u(ww')^{n-1}w, \\ r &= u(ww')^{n-1}ww', \end{aligned}$$

for some $n \geq 2$, $w' \in \Sigma^*$, and $\alpha', u, w \in \Sigma^+$.

Now let moreover $\ell \rightarrow r$ not be right s-barren, i.e., let ℓ_β be a suffix of r_α . This is expressed equivalently by the string equation $z\ell_\beta\alpha = r$ for some $z \in \Sigma^*$. Using $\ell_\beta = \alpha'$ this instantiates to

$$z\alpha'\alpha'u = u(ww')^{n-1}ww'.$$

Let $m \geq 0$ be maximal such that $((ww')^{n-1}www')^m$ is a suffix of u . Define $u_1 \in \Sigma^*$ by $u = u_1((ww')^{n-1}www')^m$. Then u_1 is a proper suffix of $(ww')^{n-1}www'$, and the equation reduces to $z\alpha'\alpha'u_1 = u_1(ww')^{n-1}www'$. If $m > 0$ then $\alpha'u_1 \in \text{OVL}(r, \ell)$, a contradiction. So $m = 0$ and $u = u_1$.

If u_1 is a suffix of ww' then $u_1w \in \text{OVL}(\ell, r)$, a contradiction. So ww' is a proper suffix of u_1 . Let $u_2 \in \Sigma^+$ be defined by $u_1 = u_2ww'$. The equation reduces to $z\alpha'\alpha'u_2 = u_2(ww')^nw$.

By definition of u_1 , u_2 is a proper suffix of $(ww')^{n-1}w$. Then $u_2 \in \text{OVL}(\ell, r)$, a contradiction. \square

If α and β do not overlap in ℓ , then we can narrow the pattern for the rule:

LEMMA 4.3. *Let $\ell \rightarrow r$ be left s-barren but not left barren. If $|\alpha| + |\beta| \leq |\ell|$ then $\ell \rightarrow r$ is of the form*

$$\alpha w x y \alpha w \rightarrow y \alpha w w x y \alpha \quad (4.2)$$

for some $x \in \Sigma^*$ and $y, \alpha, w \in \Sigma^+$.

Proof. Let $\ell \rightarrow r$ be left s-barren but not left barren. By Lemma 4.1 we get that $\ell \rightarrow r$ has the form (4.1).

Case 1: $\beta = w''(w'w)^i$ for some $0 \leq i \leq n-1$, and some non-empty suffix w'' of w . If $i \geq 1$ then $w''w \in \text{OVL}(\ell, r)$, a contradiction. So $i = 0$ and $\beta = w''$. Then

$$|r| - |\ell| = |w''| + |w| + |w'| - (|\alpha| + n|w| + (n-1)|w'|) < 0,$$

again a contradiction.

Case 2: $\beta = w''w(w'w)^i$ for some $0 \leq i \leq n-2$, and some nonempty suffix w'' of w' . If $i \geq 1$ then $w''w \in \text{OVL}(\ell, r)$, a contradiction. So $i = 0$ and $\beta = w''w$. Let $w' = xw''$ for some string x . Then we have

$$\begin{aligned} \ell &= \alpha(wxw'')^{n-1}w, \\ r &= w''w w x w'', \end{aligned}$$

and so

$$\begin{aligned} |r| - |\ell| &= 2|w''| + 2|w| + |x| - (|\alpha| + (n-1)|w''| + (n-1)|x| + n|w|) \\ &= (3-n)|w''| + (2-n)|w| + (2-n)|x| - |\alpha|. \end{aligned}$$

If $n \geq 3$ then $|r| - |\ell| < 0$. So $n = 2$ and $|r| - |\ell| = |w''| - |\alpha| > 0$ whence $|w''| > |\alpha|$. By definition of α now α is a proper suffix of w'' . Let $w'' = y\alpha$ for some $y \in \Sigma^+$. We conclude that $\ell \rightarrow r$ is of the form (4.2). \square

Putting Lemma 4.2 and 4.3 together allows us to narrow the rule pattern further:

LEMMA 4.4. *If $\ell \rightarrow r$ is left s-barren but neither left barren nor right s-barren then $\ell \rightarrow r$ is of the form*

$$\alpha w x (y \alpha w x)^{m+1} \alpha w \rightarrow y \alpha w x \alpha w w x (y \alpha w x)^{m+1} \alpha. \quad (4.3)$$

for some $m \geq 0$, $x \in \Sigma^*$, and $\alpha, w, y \in \Sigma^+$.

Proof. Let $\ell \rightarrow r$ be left s-barren but neither left barren nor right s-barren. By Lemma 4.2 we get $|\alpha| + |\beta| \leq |\ell|$. By Lemma 4.3 we get that $\ell \rightarrow r$ has the form (4.2).

The property that $\ell \rightarrow r$ is not right s-barren means that $\ell_\beta = \alpha w x$ is a suffix of $r_\alpha = y \alpha w w x y$. Then we have to solve the string equation

$$z \alpha w x = y \alpha w w x y \quad (4.4)$$

for $z, x \in \Sigma^*$, $\alpha, w, y \in \Sigma^+$.

Let $m \geq 0$ be maximal such that y^m is a suffix of x . Define $x_1 \in \Sigma^*$ by $x = x_1 y^m$. Then $z \alpha w x_1 = y \alpha w w x_1 y$ and x_1 is a proper suffix of y . Define $y_1 \in \Sigma^+$ by $y = y_1 x_1$. Then $z \alpha w = y_1 x_1 \alpha w w x_1 y_1$.

If y_1 is a suffix of w then $y_1 \in \text{OVL}(\ell, r)$, a contradiction. So w is a proper suffix of y_1 . Define $y_2 \in \Sigma^+$ by $y_1 = y_2w$. Then the equation reduces to $z\alpha = y_2wx_1\alpha wwx_1y_2$.

If y_2 is a suffix of α then $y_2w \in \text{OVL}(\ell, r)$, a contradiction. So α is a proper suffix of y_2 . Define $y_3 \in \Sigma^+$ by $y_2 = y_3\alpha$. The equation reduces to $z = y_3\alpha wx_1\alpha wwx_1y_3$ which is trivial.

By back-substitution we get

$$\begin{aligned} y &= y_1x_1 = y_2wx_1 = y_3\alpha wx_1, \\ x &= x_1y^m = x_1(y_3\alpha wx_1)^m, \\ \ell &= \alpha wx_1y\alpha w = \alpha wx_1(y_3\alpha wx_1)^{m+1}\alpha w, \\ r &= y\alpha wwx_1y\alpha = y_3\alpha wx_1\alpha wwx_1(y_3\alpha wx_1)^{m+1}\alpha. \end{aligned}$$

and thus the form (4.3) by the renaming $x_1 \mapsto x, y_3 \mapsto y$. \square

The following is interesting to note. It explains why rules of the form (4.3) were not observed by Shikishima-Tsuji et al.

THEOREM 4.5. *All rules of the form (4.3) are non-confluent.*

Proof. A one-rule SRS $\{\ell \rightarrow r\}$ where $|\ell| < |r|$ is confluent if and only if $\text{OVL}(\ell, \ell) \subseteq \text{OVL}(r, r)$ by a result of Wrathall [17]. A rule of the form (4.3) satisfies $\alpha w \in \text{OVL}(\ell, \ell)$. If $\alpha w \in \text{OVL}(r, r)$ then $\alpha w \in \text{OVL}(r, \ell)$, a contradiction to $\text{OVL}(r, \ell) = \{\alpha\}$. So $\alpha w \in \text{OVL}(\ell, \ell) \setminus \text{OVL}(r, r)$ whence $\ell \rightarrow r$ is not confluent. \square

In the next two sections we are going to identify the non-terminating and the terminating instances of the form (4.3).

5. The Non-terminating Case. A rule of the form (4.3) loops in the following case:

LEMMA 5.1. *Let $\ell \rightarrow r$ be left s -barren but neither left barren nor right s -barren. If $\ell_\beta \ell_\beta$ is a suffix of r_α , then the one-rule SRS $\{\ell \rightarrow r\}$ has a loop of length 3.*

Proof. Like in the proof of Lemma 4.1, we get $\ell_\alpha = r_\beta^{n-1}w$ and $r_\beta = ww'$ for some $w \in \Sigma^+, w' \in \Sigma^*, n \geq 2$. In the proof of Lemma 4.3 we showed $n = 2$. With $r_\alpha = v\ell_\beta \ell_\beta$ for some $v \in \Sigma^*$, we then get a loop:

$$\begin{aligned} \ell \ell_\alpha &\rightarrow r_\alpha \alpha \ell_\alpha \rightarrow r_\alpha r = v\ell_\beta \ell_\beta r_\beta \rightarrow v\ell_\beta r r_\beta = v\ell_\beta r_\beta r_\beta = v\ell r_\beta r_\beta \\ &= v\ell \ell_\alpha w'. \end{aligned}$$

\square

These loops are also instances of Kurth's criterion for loops of length 3 [9, Theorem 2, Case A]. The following little result provides an alternative criterion to Lemma 5.1.

LEMMA 5.2. *If $\ell \rightarrow r$ has the form (4.3) then the following are equivalent:*

1. $\ell_\beta \ell_\beta$ is a suffix of r_α ,
2. $m = 0$ and $y = y'\alpha wx$ for some $y' \in \Sigma^+$.

Proof. Obviously (2) implies (1). Next we show the converse by contradiction. Let $\ell \rightarrow r$ have the form (4.3) and let $\ell_\beta \ell_\beta$ be a suffix of r_α . Define $v \in \Sigma^*$ by $r_\alpha = v\ell_\beta \ell_\beta$. If $m > 0$ then y is a suffix of $y\alpha w$ and then $y\alpha w \in \text{OVL}(\ell, r)$, a contradiction. With $m > 0$, the string αwx is a suffix of αwwx_1y . If y is a suffix of αwx then $y\alpha w \in \text{OVL}(\ell, r)$, a contradiction. So αwx is a proper suffix of y , i.e., there is $y' \in \Sigma^+$ such that $y = y'\alpha wx$. \square

EXAMPLE 1. *The one-rule SRS*

$$abdababab \rightarrow dabababbdababa$$

has a loop of length 3:

$$\begin{aligned}
&\underline{abdabababbdababab} \rightarrow \\
&dabababbdababab\underline{abdababab} \rightarrow \\
&dabababbdabababbdababab \rightarrow \\
&dabababbdabababab\underline{abdabababbdababab} dababab.
\end{aligned}$$

Redexes are underlined. The re-occurrence of the start string is indicated by a box. This example provides the smallest non-terminating witness ($|r| = 14$) of Lemma 4.4.

6. The Terminating Case. For this section let us assume a rule of the form (4.3) where $\ell_\beta \ell_\beta$ is not a suffix of r_α . We are going to reduce termination of such a rule to termination of an SRS R over a different alphabet. Termination of R will be easy to prove.

Define r_δ , $r_{\beta,\alpha}$, and $r_{\beta,\delta}$ by

$$r = r_\delta \ell_\beta \alpha, \quad r = \beta r_{\beta,\alpha} \alpha, \quad r = \beta r_{\beta,\delta} \ell_\beta \alpha.$$

These definitions are sound as witnessed by

$$\begin{aligned}
\beta &= y\alpha wx\alpha w, \\
\ell_\beta &= \alpha wx(y\alpha wx)^m, \\
r_\delta &= y\alpha wx\alpha wxy, \\
r_{\beta,\alpha} &= wx(y\alpha wx)^{m+1}, \\
r_{\beta,\delta} &= wxy.
\end{aligned}$$

LEMMA 6.1. Let $\ell \rightarrow r$ have the form (4.3). Then the following rewrite steps exist:

$$\begin{aligned}
r_\alpha r &\rightarrow_{\ell \rightarrow r} r_\delta r r_\beta, & r_\alpha r_\alpha &\rightarrow_{\ell \rightarrow r} r_\delta r r_{\beta,\alpha}, & r_\alpha r_\delta &\rightarrow_{\ell \rightarrow r} r_\delta r r_{\beta,\delta}, \\
r_{\beta,\alpha} r &\rightarrow_{\ell \rightarrow r} r_{\beta,\delta} r r_\beta, & r_{\beta,\alpha} r_\alpha &\rightarrow_{\ell \rightarrow r} r_{\beta,\delta} r r_{\beta,\alpha}, & r_{\beta,\alpha} r_\delta &\rightarrow_{\ell \rightarrow r} r_{\beta,\delta} r r_{\beta,\delta}.
\end{aligned}$$

Proof. Routine. \square

LEMMA 6.2. Let $\ell \rightarrow r$ have the form (4.3) and let $\ell_\beta \ell_\beta$ not be a suffix of r_α . Then ℓ is not a factor of any of the following: (1) $r_\delta^i r$, (2) $r r_\beta$, (3) $r r_{\beta,\delta} r_\delta^i r$ for any $i \geq 0$.

Proof. For Claim 1, let $i \geq 1$ be least such that ℓ is a factor of $r_\delta^i r$. Then ℓ_β is a suffix of r_δ^i because β is the only overlap of ℓ with r . Since $\ell_\beta \ell_\beta$ is not a suffix of $r_\alpha = r_\delta \ell_\beta$, ℓ_β is not a suffix of y . Hence y is a proper suffix of ℓ_β and so of $y\alpha wx$. So $y\alpha w \in \text{OVL}(\ell, r)$, a contradiction.

For Claim 2, let ℓ be a factor of $r r_\beta$. Because α is the only overlap between r and ℓ , we have $|\ell_\alpha| \leq |r_\beta|$, a contradiction.

For Claim 3 assume that ℓ is a factor of $r r_{\beta,\delta} r_\delta^i r$ for some $i \geq 0$. By Claims 1 and 2, ℓ is neither a factor of $r_{\beta,\delta} r_\delta^i r$ nor of $r r_{\beta,\delta}$; so ℓ is of the form $\ell' r_{\beta,\delta} r_\delta^j \ell''$ for some $0 \leq j \leq i$ and some non-empty suffix ℓ' of r and some non-empty prefix ℓ'' of r . Thus ℓ is of the form $\alpha r_{\beta,\delta} r_\delta^j \beta$. If $j = 0$ then $wx(y\alpha wx)^m = wxy$ which contradicts $y, \alpha \in \Sigma^+$. So $j > 0$ and y is a proper suffix of ℓ_β . We get a contradiction by $y\alpha w \in \text{OVL}(\ell, r)$.

\square

The six-rule SRS R over $\Omega = \{a, b, c, d, e, f\}$ is defined as follows:

$$\begin{aligned}
R = \{ &g'g'' \rightarrow h'fh'' \mid (g', h') \in \{(a, d), (c, e)\}, \\
&(g'', h'') \in \{(a, c), (d, e), (f, b)\} \}
\end{aligned}$$

Define the weight $wt^*(x)$ of a string x by $wt(a) = wt(c) = 3$, $wt(b) = wt(d) = wt(e) = wt(f) = 1$, and $wt^*(x_1 \dots x_k) = \sum_{i=1}^k wt(x_i)$. Then R terminates by

$$wt^*(u) - wt^*(v) = (wt(g') - wt(h')) - wt(f) + (wt(g'') - wt(h'')) = 2 - 1 + 0 > 0$$

for all rewrite steps $u \rightarrow_R v$.

Let the string homomorphism $\phi : \Omega^* \rightarrow \Sigma^*$ be defined by $\phi(a) = r_\alpha, \phi(b) = r_\beta, \phi(c) = r_{\beta,\alpha}, \phi(d) = r_\delta, \phi(e) = r_{\beta,\delta}, \phi(f) = r$. By Lemma 6.1, $u \rightarrow_R v$ implies $\phi(u) \rightarrow_{\ell \rightarrow r} \phi(v)$ for all $u, v \in \Omega^*$. However we will need the converse direction. To this end let us define the regular language \mathcal{M} by

$$\mathcal{M} = (a + d(fe)^* + d(fe)^*fc)^*(af + d(fe)^*f(cf + b)) + f.$$

Let $\phi[\mathcal{M}]$ denote the set $\{\phi(u) \mid u \in \mathcal{M}\}$. We are going to show that $\{\ell \rightarrow r\}$ -reduction steps on $\phi[\mathcal{M}]$ can be simulated by R -reduction steps. First we show that R -reduction preserves $\phi[\mathcal{M}]$.

LEMMA 6.3. *If $u \in \mathcal{M}$ and $u \rightarrow_R v$ then $v \in \mathcal{M}$.*

Proof. Let $(g', h') \in \{(a, d), (c, e)\}$ and $(g'', h'') \in \{(a, c), (d, e), (f, b)\}$. Let $u = u'g'g''u'' \in \mathcal{M}$ and $v = u'h'fh''u''$. Then we derive

$$\begin{aligned} u' &\in (a + d(fe)^* + d(fe)^*fc)^* & \text{if } g' = a, \\ u' &\in (a + d(fe)^* + d(fe)^*fc)^*d(fe)^*f & \text{if } g' = c. \end{aligned}$$

Case 1: $g'' = a$. If $g' = a$ then $u'' \in \mathcal{M}$ whence $v = u'dfcu'' \in \mathcal{M}$. If $g' = c$ then $u'' = f$ whence $v = u'efcu'' \in \mathcal{M}$.

Case 2: $g'' = d$. Then

$$\begin{aligned} u'' &\in ((fe)^* + (fe)^*fc)(a + d(fe)^* + d(fe)^*fc)^*(af + d(fe)^*f(cf + b)) \\ &\quad + (fe)^*f(cf + b). \end{aligned}$$

If $g' = a$ then $v = u'dfeu'' \in \mathcal{M}$. If $g' = c$ then $v = u'efeu'' \in \mathcal{M}$.

Case 3: $g'' = f$. If $g' = a$ then u'' is the empty string and $v = u'dfbu'' \in \mathcal{M}$. If $g' = c$ then u'' is again the empty string and $v = u'efbu'' \in \mathcal{M}$. \square

Next we derive a few properties of $u \in \mathcal{M}$ if $\phi(u)$ contains a factor ℓ .

LEMMA 6.4. *Let $u \in \mathcal{M}$ and $s', s'' \in \Sigma^*$. If $\phi(u) = s'\ell s''$ then $u = u'g'g''u''$, $|\phi(u')| \leq |s'| < |\phi(u'g')|$, $|\phi(u'')| \leq |s''| < |\phi(g''u'')|$ for some $u', u'' \in \Omega^*$, $g' \in \{a, c\}$, $g'' \in \{a, d, f\}$.*

Proof. Suppose that $u \in \mathcal{M}$, $s', s'' \in \Sigma^*$, and $\phi(u) = s'\ell s''$. Let $u' \in \Omega^*$ be the longest prefix of u such that $|\phi(u')| \leq |s'|$. Let $u'' \in \Omega^*$ be the longest suffix of u such that $|\phi(u'')| \leq |s''|$. By $|\phi(u)| > |\phi(u'u'')|$ there is $v \in \Sigma^+$ such that $u = u'vu''$. Define $t', t'' \in \Sigma^*$ by $s' = \phi(u')t'$ and $s'' = t''\phi(u'')$. Then

$$\phi(u) = \phi(u')\phi(v)\phi(u'') = \phi(u')t'\ell t''\phi(u''),$$

whence $\phi(v) = t'\ell t''$. The case $|v| = 1$ implies that ℓ is a factor of r , so $|v| \geq 2$. We distinguish cases on the form of v .

Case 1: $v \in \Omega^*(a + c)(a + d + f)\Omega^*$. Let $g' \in \{a, c\}$, $g'' \in \{a, d, f\}$, $v', v'' \in \Omega^*$, and let $v = v'g'g''v''$. We further distinguish cases whether v', v'' are empty strings or not.

Case 1.1: $|v'| = |v''| = 0$. Then $v = g'g''$. By definition of u' we get $|t'| < |\phi(g')|$. By definition of u'' we get $|t''| < |\phi(g'')|$. The claim follows.

Case 1.2: $|v'| = 0, |v''| > 0$. By $|r| > |\ell|$ and $|r_\alpha| > |\ell|$ and $u \in \mathcal{M}$ we get $v \in (a+c)d^+(a+d+f)$. Let $v = v_0g_0$ for some $v_0 \in (a+c)d^+$, and $g_0 \in \{a,d,f\}$. Then there are $\ell', \ell'' \in \Sigma^+$ such that $\ell = \ell'\ell''$, $\phi(v_0) = t'\ell'$, and $\phi(g_0) = \ell''t''$. Since $\phi(g_0)$ is a prefix of r , we obtain $\ell'' \in \text{OVL}(\ell, r)$, so $\ell'' = \beta$ and $\ell' = \ell_\beta$. By definition of v_0 , now $\phi(d) = r_\delta = y\alpha wx\alpha wxy$ is a suffix of $\ell_\beta = \alpha wx(y\alpha wx)^m$. So $m > 0$ and y is a suffix of $y\alpha wx$. Then $y\alpha w \in \text{OVL}(\ell, r)$, a contradiction.

Case 1.3: $|v'| > 0, |v''| = 0$. Let $v = v_0g_0$ for some $v_0 \in \Omega^+(a+c)$, and $g_0 \in \{a,d,f\}$. Then there are $\ell', \ell'' \in \Sigma^+$ such that $\ell = \ell'\ell''$, $\phi(v_0) = t'\ell'$, and $\phi(g_0) = \ell''t''$. Since $\phi(g_0)$ is a prefix of r , we obtain $\ell'' \in \text{OVL}(\ell, r)$, so $\ell'' = \beta$ and $\ell' = \ell_\beta$. Then

$$|\ell_\beta| = |\phi(v_0)| > |\phi(c)| = |r_{\beta,\alpha}| > |\ell_\beta|,$$

a contradiction.

Case 1.4: $|v'|, |v''| > 0$. By $|r| > |\ell|$ and $|r_\alpha| > |\ell|$ and $u \in \mathcal{M}$ we get $g' = c$ and $g'' = d$. So $\phi(cd) = r_{\beta,\alpha}r_\delta$ is a factor of ℓ , whence $|r_{\beta,\alpha}r_\delta| \leq |\ell|$, a contradiction.

Case 2: $v \in \Omega^+ \setminus \Omega^*(a+c)(a+d+f)\Omega^*$. Define the set of fragments $\mathcal{F}(z)$ of a string $z \in \Omega^*$ as follows. If $z \in (\Omega \setminus \{f\})^*$ then $\mathcal{F}(z) = \{z\}$. Else $z = z_0fz_1 \dots fz_n$ for some $n \geq 1$ and unique $z_1, \dots, z_n \in (\Omega \setminus \{f\})^*$; then

$$\mathcal{F}(z) = \{z_1f, fz_2f, \dots, fz_{n-1}f, fz_n\}.$$

From $u \in \mathcal{M}$ then

$$\mathcal{F}(u) \in (a+d)^*f + f(e+c)(a+d)^*f + fb.$$

Because $|r| > |\ell|$, and ℓ is not a factor of r , we obtain $v \in \mathcal{F}(u)$. So

$$v \in \mathcal{F}(u) \setminus \Omega^*(a+c)(a+d+f)\Omega^* = d^*f + fed^*f + fb.$$

By Lemma 6.2, $\phi(v)$ has no factor ℓ , so this case is void. \square

Now we are ready to state the simulation lemma.

LEMMA 6.5. *Let $u \in \mathcal{M}$ and $t \in \Sigma^*$. If $\phi(u) \rightarrow_{t \rightarrow r} t$ then $\phi(v) = t$ and $u \rightarrow_R v$ for some $v \in \mathcal{M}$.*

Proof. Let $u \in \mathcal{M}$ and $s', s'', t \in \Sigma^*$, and let $\phi(u) = s'\ell s''$ and $t = s'rs''$. By Lemma 6.4 there are $u', u'' \in \Omega^*$, $g' \in \{a, c\}$, $g'' \in \{a, d, f\}$ such that $u = u'g'g''u''$ and $|\phi(u')| \leq |s'| < |\phi(u'g')|$ and $|\phi(u'')| \leq |s''| < |\phi(g''u'')|$. Define $t', t'' \in \Sigma^*$ by $s' = \phi(u')t'$ and $s'' = t''\phi(u'')$. Then

$$\phi(u) = \phi(u')\phi(g')\phi(g'')\phi(u'') = \phi(u')t'\ell t''\phi(u''),$$

so $\phi(g')\phi(g'') = t'\ell t''$. By $|s''| < |\phi(g''u'')|$ we get $|t''| < |\phi(g'')|$. Define $\ell'' \in \Sigma^+$ by $\phi(g'') = \ell''t''$. Define $\ell' \in \Sigma^+$ by $\ell = \ell'\ell''$. So $\phi(g') = t'\ell'$. By $|s'| < |\phi(u'g')|$ we get $|t'| < |\phi(g')|$ and so $\ell' \in \Sigma^+$.

Since $\phi(g'')$ is a prefix of r , we obtain $\ell'' \in \text{OVL}(\ell, r)$, so $\ell'' = \beta$ and $\ell' = \ell_\beta$. Define $h', h'' \in \Omega$ by

$$h' = \begin{cases} d & \text{if } g' = a, \\ e & \text{if } g' = c, \end{cases} \quad h'' = \begin{cases} c & \text{if } g'' = a, \\ e & \text{if } g'' = d, \\ b & \text{if } g'' = f. \end{cases}$$

Then $g'g'' \rightarrow h'fh''$ is in R , and moreover $\phi(g') = \phi(h')\ell_\beta = t'\ell_\beta$ and $\phi(g'') = \beta\phi(h'') = \beta t''$. So $t' = \phi(h')$ and $t'' = \phi(h'')$ and so

$$t = s'rs'' = \phi(u')\phi(h')\phi(f)\phi(h'')\phi(u'') = \phi(v)$$

for $v = u'h'fh''u''$. So $u \rightarrow_R v$. By Lemma 6.3 we get $v \in \mathcal{M}$. \square

We are about to prove termination of $\ell \rightarrow r$ by a reduction to termination of R . For this purpose we still need $\{\ell \rightarrow r\}$ -reductions that start in $\phi[\mathcal{M}]$. Such reductions are provided by forward closures [10, 1] as we will show next. We use the following characterization of forward closures by Hermann.

DEFINITION 6.6 ([6, Corollaire 2.16]). *The set of forward closures of a string rewriting rule $\ell \rightarrow r$ over alphabet Σ is the least set $\text{FC}(\ell \rightarrow r)$ of $\ell \rightarrow r$ -reductions such that*

- fc1. $(\ell \rightarrow r) \in \text{FC}(\ell \rightarrow r)$,
- fc2. if $(s_1 \rightarrow^+ t_1' \ell') \in \text{FC}(\ell \rightarrow r)$ and $\ell = \ell' \ell''$ for some $\ell', \ell'' \in \Sigma^+$ then $(s_1 \ell'' \rightarrow^+ t_1' \ell' \ell'' \rightarrow^+ t_1' r) \in \text{FC}(\ell \rightarrow r)$,
- fc3. if $(s_1 \rightarrow^+ t_1' \ell t_1'') \in \text{FC}(\ell \rightarrow r)$ then $(s_1 \rightarrow^+ t_1' \ell t_1'' \rightarrow^+ t_1' r t_1'') \in \text{FC}(\ell \rightarrow r)$.

LEMMA 6.7. *Every forward closure of a rule $\ell \rightarrow r$ of the form (4.3) where $\ell_\beta \ell_\beta$ is not a suffix of r_α , has a right hand side in $\phi[\mathcal{M}]$.*

Proof. By induction along the definition of forward closure. Let $(s \rightarrow^+ t) \in \text{FC}(\ell \rightarrow r)$. In Case (fc1) we have $t = r = \phi(f)$. In Case (fc3) the claim follows from Lemma 6.5. This leaves to prove Case (fc2).

Suppose that $s = s_1 \ell''$, $t = t_1' r$, $(s_1 \rightarrow^+ t_1' \ell') \in \text{FC}(\ell \rightarrow r)$, and $\ell = \ell' \ell''$ for some $\ell', \ell'' \in \Sigma^+$. By inductive hypothesis, there is $u \in \mathcal{M}$ such that $t_1' \ell' = \phi(u)$. By definition of \mathcal{M} , u has suffix f or fb .

Case 1: u has suffix fb . Define $g' \in \Omega^*$ by $u = g' fb$. Then

$$g' \in (a + d(fe)^* + d(fe)^* fc)^* d(fe)^*$$

by definition of \mathcal{M} . We distinguish cases whether $|\ell'| > |r_\beta|$ or not.

Case 1.1: $|\ell'| > |r_\beta|$. The string $t_1' \ell'$ has suffix $\phi(fb) = rr_\beta$. By $|\ell| < |r|$ and $|\ell'| > |r_\beta|$ we get $\ell' = zr_\beta$ for some non-empty suffix z of r . Now $z \in \text{OVL}(r, \ell)$, so $z = \alpha$. So $t_1' \ell' = \phi(g') rr_\beta = \phi(g') r_\alpha \ell'$, whence $t_1' = \phi(g') r_\alpha = \phi(g' a)$. So $t_1' r = \phi(g' a) r = \phi(g' a f)$ for $g' a f \in \mathcal{M}$.

Case 1.2: $|\ell'| \leq |r_\beta|$. Then ℓ' is a suffix of r_β and so of r . So $\ell' \in \text{OVL}(r, \ell)$ whence $\ell' = \alpha$. So $t_1' \ell' = \phi(g' f) r_\beta = \phi(g' f) r_{\beta, \alpha} \ell'$, whence $t_1' = \phi(g' f) r_{\beta, \alpha} = \phi(g' fc)$. So $t_1' r = \phi(g' fc) r = \phi(g' fcf)$ for $g' fcf \in \mathcal{M}$.

Case 2: u has suffix f . Define $g' \in \Omega^*$ by $u = g' f$. Then

$$g' \in (a + d(fe)^* + d(fe)^* fc)^*$$

by definition of \mathcal{M} . By $|\ell| < |r|$ we get that $\ell' \in \text{OVL}(r, \ell)$, whence $\ell' = \alpha$. So $t_1' \ell' = \phi(g' f) = \phi(g') r = \phi(g') r_\alpha \ell'$, whence $t_1' = \phi(g') r_\alpha = \phi(g' a)$. So $t_1' r = \phi(g' a) r = \phi(g' a f)$ for $g' a f \in \mathcal{M}$. \square

LEMMA 6.8. *A rule $\ell \rightarrow r$ of the form (4.3) terminates if $\ell_\beta \ell_\beta$ is not a suffix of r_α .*

Proof. If $\ell \rightarrow r$ is non-terminating then there is an infinite rewriting sequence $s_1 \rightarrow_{\ell \rightarrow r} s_2 \rightarrow_{\ell \rightarrow r} \dots$ starting from a right hand side of a forward closure [1]. By Lemma 6.7 $s_1 \in \phi[\mathcal{M}]$, i.e., there is $u_1 \in \mathcal{M}$ such that $\phi(u_1) = s_1$. By induction on i , using Lemma 6.5, one easily proves that for every i there is an $u_{i+1} \in \mathcal{M}$ such that both $u_i \rightarrow_R u_{i+1}$ and $\phi(u_{i+1}) = s_{i+1}$. Hence we get an infinite reduction sequence $u_1 \rightarrow_R u_2 \rightarrow_R \dots$. Contradiction to termination of R . \square

EXAMPLE 2. *For every $m \geq 0$, the one-rule SRS*

$$ab(dab)^{m+1}ab \rightarrow dababb(dab)^{m+1}a$$

is terminating by Lemma 6.8. With $m = 0$ we get the smallest terminating witness ($|r| = 10$) of Lemma 4.4.

This example also proves that Kurth's [8] Criterion F is incomplete, for Criterion F applies only to the left barren or right barren cases [3, Theorem 6.31].

We note moreover that the maximal length of a derivation starting with $s \in \Sigma^*$ is linear in $|s|$. This is a direct consequence of the decreasing weight associated with a step $u \rightarrow_R v$.

7. The Main Theorem. Now we have all material together to prove our claim.

THEOREM 7.1. *Let $|\text{OVL}(r, \ell)| = |\text{OVL}(\ell, r)| = 1$. Then $\{\ell \rightarrow r\}$ terminates if and only if it has no loop of lengths 1, 2, or 3.*

Proof. Let $\text{OVL}(r, \ell) = \{\alpha\}$ and $\text{OVL}(\ell, r) = \{\beta\}$. If ℓ is a factor of r then $\{\ell \rightarrow r\}$ has a loop of length 1 [8]. Else if $|\ell| \geq |r|$ then $\{\ell \rightarrow r\}$ terminates. If $\ell \rightarrow r$ is left barren or right barren then $\{\ell \rightarrow r\}$ terminates. So suppose that ℓ is not a factor of r ; that $|\ell| < |r|$; and that $\ell \rightarrow r$ is neither left barren nor right barren. We distinguish cases:

Case 1: $\ell \rightarrow r$ is neither left s-barren nor right s-barren. Then $r = r'\ell_\beta\alpha$ and $r = \beta\ell_\alpha r''$ for some strings r', r'' . There is a loop of length 2:

$$\ell\ell_\alpha \rightarrow r\ell_\alpha = r'\ell_\beta\alpha\ell_\alpha = r'\ell_\beta\ell \rightarrow r'\ell_\beta r = r'\ell_\beta\beta\ell_\alpha r'' = r'\ell_\alpha r''.$$

Case 2: $\ell \rightarrow r$ is left s-barren but not right s-barren. Then $\ell \rightarrow r$ has the form (4.3). If $\ell_\beta\ell_\beta$ is a suffix of r_α then $\{\ell \rightarrow r\}$ has a loop of length 3 by Lemma 5.1. Else $\{\ell \rightarrow r\}$ terminates by Lemma 6.8.

Case 3: $\ell \rightarrow r$ is not left s-barren but right s-barren. This case is symmetric to Case 2: We have a loop of length 3 if $\ell_\alpha\ell_\alpha$ is a prefix of r_β , otherwise termination.

Case 4: $\ell \rightarrow r$ is both left s-barren and right s-barren. Then Lemma 4.1 and its dual apply, showing $|\beta| > |\alpha|$ and $|\alpha| > |\beta|$, a contradiction. So this case does not exist. This finishes the proof. \square

Kurth [9] has proved decidability of the existence of loops of lengths 1, 2, or 3 for one-rule SRSs. Indeed, for every SRS and every $n \geq 1$, the existence of loops of lengths less or equal n is decidable [5].

COROLLARY 7.2. *Termination is decidable for one-rule SRSs $\{\ell \rightarrow r\}$ that satisfy $|\text{OVL}(r, \ell)| = |\text{OVL}(\ell, r)| = 1$.*

8. Conclusion. We proved that termination of one-rule SRSs with one pair of overlaps is equivalent to the non-existence of loops of length less than or equal to 3. Thus we showed that termination is decidable for one-rule SRSs with one pair of overlaps. A surprising observation in this investigation was the emergence of non-tame rules, some admitting loops of length 3, and some terminating. Such rules were not covered by the two precursor results by Kurth and by Shikishima-Tsuji et al.

Acknowledgements. Robert McNaughton gave the author an appreciation of the intricacy of the problem.

REFERENCES

- [1] N. DERSHOWITZ, *Termination of linear rewriting systems*, in Proc. 8th Int. Coll. Automata, Languages and Programming, LNCS 115, Springer, 1981, pp. 448–458.
- [2] A. GESER, *Decidability of termination of grid string rewriting rules*, SIAM J. Comput., 31 (2002), pp. 1156–1168.
- [3] ———, *Is termination decidable for string rewriting with only one rule?*, habilitation thesis, Wilhelm-Schickard-Institut, Universität Tübingen, Germany, Jan. 2002. 201 pages.
- [4] ———, *Loops of superexponential lengths in one-rule string rewriting*, in Proc. 13th Int. Conf. Rewriting Techniques and Applications, S. Tison, ed., LNCS 2378, Springer, 2002, pp. 267–280.
- [5] A. GESER AND H. ZANTEMA, *Non-looping string rewriting*, Theoret. Informatics Appl., 33 (1999), pp. 279–301.

- [6] M. HERMANN, *Divergence des systèmes de réécriture et schématisation des ensembles infinis de termes*, habilitation, Université de Nancy, France, Mar. 1994.
- [7] Y. KOBAYASHI, M. KATSURA, AND K. SHIKISHIMA-TSUJI, *Termination and derivational complexity of confluent one-rule string rewriting systems*, Theoret. Comput. Sci., 262 (2001), pp. 583–632.
- [8] W. KURTH, *Termination und Konfluenz von Semi-Thue-Systemen mit nur einer Regel*, dissertation, Technische Universität Clausthal, Germany, 1990.
- [9] —, *One-rule semi-Thue systems with loops of length one, two, or three*, RAIRO Inform. Théor., 30 (1995), pp. 415–429.
- [10] D. S. LANKFORD AND D. R. MUSSER, *A finite termination criterion*, tech. rep., Information Sciences Institute, Univ. of Southern California, Marina-del-Rey, CA, 1978.
- [11] Y. MATIYASEVITCH AND G. SÉNIZERGUES, *Decision problems for semi-Thue systems with a few rules*, in Proc. 11th IEEE Symp. Logic in Computer Science, New Brunswick, NJ, July 1996, IEEE Computer Society Press, pp. 523–531.
- [12] R. MCNAUGHTON, *The uniform halting problem for one-rule Semi-Thue Systems*, Tech. Rep. 94-18, Dept. of Computer Science, Rensselaer Polytechnic Institute, Troy, NY, Aug. 1994. See also “Correction to ‘The Uniform Halting Problem for One-rule Semi-Thue Systems’”, unpublished paper, Aug., 1996.
- [13] —, *Well-behaved derivations in one-rule Semi-Thue Systems*, Tech. Rep. 95-15, Dept. of Computer Science, Rensselaer Polytechnic Institute, Troy, NY, Nov. 1995. See also “Correction by the author to ‘Well-behaved derivations in one-rule Semi-Thue Systems’”, unpublished paper, July, 1996.
- [14] —, *Semi-Thue Systems with an Inhibitor*, J. Automated Reasoning, 26 (1997), pp. 409–431.
- [15] G. SÉNIZERGUES, *On the termination problem for one-rule Semi-Thue Systems*, in Proc. 7th Int. Conf. Rewriting Techniques and Applications, H. Ganzinger, ed., LNCS 1103, Springer, 1996, pp. 302–316.
- [16] K. SHIKISHIMA-TSUJI, M. KATSURA, AND Y. KOBAYASHI, *On termination of confluent one-rule string rewriting systems*, Inform. Process. Lett., 61 (1997), pp. 91–96.
- [17] C. WRATHALL, *Confluence of one-rule Thue systems*, in Word Equations and Related Topics, K. U. Schulz, ed., LNCS 572, Springer, 1992.
- [18] H. ZANTEMA AND A. GESER, *A complete characterization of termination of $0^p1^q \rightarrow 1^r0^s$* , Applicable Algebra in Engineering, Communication, and Computing, 11 (2000), pp. 1–25.

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.				
1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE October 2002	3. REPORT TYPE AND DATES COVERED Contractor Report		
4. TITLE AND SUBTITLE TERMINATION OF STRING REWRITING RULES THAT HAVE ONE PAIR OF OVERLAPS		5. FUNDING NUMBERS C NAS1-97046 WU 505-90-52-01		
6. AUTHOR(S) Alfons Geser				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) ICASE Mail Stop 132C NASA Langley Research Center Hampton, VA 23681-2199		8. PERFORMING ORGANIZATION REPORT NUMBER ICASE Report No. 2002-33		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Langley Research Center Hampton, VA 23681-2199		10. SPONSORING/MONITORING AGENCY REPORT NUMBER NASA/CR-2002-211923 ICASE Report No. 2002-33		
11. SUPPLEMENTARY NOTES Langley Technical Monitor: Dennis M. Bushnell Final Report To be submitted to STACS 2002.				
12a. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified-Unlimited Subject Category 60, 61 Distribution: Nonstandard Availability: NASA-CASI (301) 621-0390		12b. DISTRIBUTION CODE		
13. ABSTRACT (Maximum 200 words) This paper presents a partial solution to the long standing open problem of termination of one-rule string rewriting. Overlaps between the two sides of the rule play a central role in existing termination criteria. We characterize termination of all one-rule string rewriting systems that have one such overlap at either end. This both completes a result of Kurth and generalizes a result of Shikishima-Tsuji et al.				
14. SUBJECT TERMS semi-Thue system, string rewriting, one-rule, single-rule, termination, uniform termination, overlap			15. NUMBER OF PAGES 16	
			16. PRICE CODE A03	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT	